and reference is made by figures or letters. This method leaves the drawing free from writing, an advantage which does not require any comment.

The aim of the course should determine the kind of drawings to be made, and I think that it is clear from the foregoing statements just what the pupil should do in the way of drawings and how he should do it. Clearness and strict adherence to the fact are indispensable. Every line should mean something. In Agassiz's laboratory, which stood on Penekese Island, a class of students, a number of whom have become noted as biologists, worked with this great naturalist. On the walls were a few mottoes, for Agassiz believed in mottoes. One of these now hangs in the Marine Biological Laboratory at Wood's Hole, Mass. Agassiz realized the value of drawing to the scientist when he placed this motto before his students: "The pencil is the best of eyes."

TO WHAT EXTENT IS A CLOSER CORRELATION OF THE DIFFERENT BRANCHES OF COLLEGE MATHEMATICS DESIRABLE FROM THE TEACHER'S STANDPOINT.*

By W. H. Wilson,

Professor of Mathematics, University of Wooster.

Considering the vital point at issue as brought out in recent papers suggested by the Perry movement, I feel justified in making the term "college mathematics" include both pure and applied mathematics. The teacher's desire is, of course, what he believes will best equip the student for the kind of mental activity in which he will be engaged.

Most of the students of the present day who study college mathematics do not do so primarily for its own peculiar mental discipline. It is becoming more and more evident that a majority of those who are now electing courses of pure mathematics, do so because they desire equipment for specific kinds of work in departments where higher mathematics is indispensable.

The condition was different, at least in the middle West, a

* Read before the Association of Ohio Teachers of Mathematics and Science, Columbus, Ohio, December 30, 1904.

generation ago. Few of our college, or scientific, men pursued scientific or engineering work to a point where they made any use of calculus. The available courses were limited. The student with vigorous reasoning powers pursued higher mathematics as a kind of mental gymnastics and was satisfied with the general culture and power received. In a majority of cases he made no direct use of the results. The changed attitude here referred to is due to two causes:

First, the development of the profession of engineering, which to-day is so inviting to the young man, has been quite rapid. The engineer and the scientist are both finding higher mathematics to be an important part of his necessary equipment, which he must not only possess, but which he must know how to use with ease in getting reliable results.

Second, other courses have been so multiplied that the student who does not expect to use higher mathematics, in most cases, finds other courses more inviting, and, if well chosen, more profitable.

The result has not been a loss of prestige to the department of mathematics, except in the minds of instructors who have clung to old ideals in the face of radically changed conditions. I do not pretend to say that the pursuit of the higher forms of abstract mathematics, where the specialist reaches conclusions which he himself cannot connect with anything real, much less with any form of practical utility, shall be abandoned. Non-Euclidean geometry possesses little of human interest and is easily set aside as a recreation for the specialist. It would not be surprising if, some time in some way entirely unforseen, abstract studies in higher mathematics such as non-Euclidean geometry should become useful. Greater surprises have occurred in the realm of science. That time is not yet. Undergraduate courses have little if any room for such. If they were offered they would usually lack undergraduate students. Graduate courses have almost the entire field here.

This development in modern education is a very natural one, and is rather to be invited than deplored. It is the business of the mathematician to touch as many points of human interest as it is possible for him to touch. If we are quick to seize the opportunity a larger field of usefulness awaits us. One of the noblest characters I ever knew had as her motto "I serve." She lived in accordance with her motto, too, but there was nothing in this attitude of mind which detracted from her dignity, and certainly not from her worth.

Let us look a little into the methods by which we are to meet these conditions. Much should be done before the student enters college, but the responsibility rests no less heavily upon the college teacher, for in most cases he trains the teacher who trains the entering collegian. Throughout his mathematical studies the student's theories should be connected in a vital way with the other fields of his mental activity, and, if necessary to the accomplishment of this end, the pruning knife must be applied till the more productive branches have room to develop and bring their fruit to a fuller maturity. Not only will the tree be more healthy and beautiful, but the fruit will be more appetizing as well as more nourishing.

Graphic methods not only lead the student naturally into exact measurement, and into the processes of mechanical drawing, but they also give a more comprehensive as well as a detailed view of the relations studied, because they utilize the experienced visual sense. Graphical methods also place the best possible check on the analytic processes.

In talking with a student of a leading technical school I was surprised to learn the extent to which they depend upon graphics in practical work where considerable precision is necessary. The student must always have the analytic method in reserve, but the necessity of its constant use is relieved by drawing to scale. My attention was first called to the possibilities of mechanical drawing in a very forcible way a few years ago. A friend of mine, who is a manufacturer and an inventor, had designed a new model for the mould-board of a plow. He had no technical knowledge of mathematics, not even of the names of the trigonometric ratios; yet he had chosen a geometric surface which I had never met in my work. He employed me to help him get such command of the surface that he might be able to meet certain conditions. In technical language the problem was so to modify the parameters of the surface as to accomplish a desired end. I attacked the problem in accordance with my training by the analytic method. The equation of the surface was easily gotten. The plan of operation was easily outlined. But the computations which I deemed necessary were very laborious. After several days of pretty hard work I showed him my results. It was impossible to carry him through my work, so I simply gave him the results with specific directions how to accomplish results within the limited range of my computation. He accepted my work and listened to my explanation of some of the geometric properties of the surface. On the following day I called to make some additional explanations. To my astonishment he had, by robbing his time for sleep, taken up the matter, from my explanations, entirely independently of my computations, and had graphically solved the problem in a very simple way with an accuracy far beyond his needs. You may be sure I never forgot the lesson.

Teachers of college mathematics and their colleagues in the secondary schools ought to agree upon a line of action which will result in a thorough unification of the analytic with the graphic, applied to problems borrowed from some fields of science or art, in the mathematics of all secondary schools. Drawing to scale will often give a student control over an analytic situation which otherwise he would grasp only partially if at all. The time allotted to mathematics need not be increased. Sufficient time could be gained by removing from our mathematical courses some of the rubbish which has accumulated from past ages. The result would be that students entering college would be at home from the very start in the methods which the practical scientist and engineer must have.

What, then, shall the teacher of college mathematics do under the improved conditions here outlined? The correlation here sought is not to be found in some internal bond of union, but in that application to the things of real human interest, which cause the student to lose sight of the exact classification of the processes in his eagerness to reach the conclusion not simply for its own sake, but for its bearing on some investigation the data for which come from some source other than a fertile imagination.

The most effective means of the desired correlation are to be sought in the mind of the instructor. He must, of course, have a training in pure mathematics much beyond the subject he is trying to teach, in order that he may guide the student along the lines of least resistance and greatest utility. He must draw aside the veil here and there and display vistas which the student may glance at in their proper relations, but must not enter at the time. The instructor must not only have this advanced training in pure mathematics, but he must be at home in the more important branches of applied mathematics, in order that he may draw upon them for illustrations and carry on investigations with an appreciation of the exact bearings of his mathematical conclusions. I firmly believe that he should have at least one hobby in applied mathematics in the riding of which he may get that exercise which makes him virile and interested in human affairs. Surely by following such a course an instructor is in a better position to help the present day student than if he were to spend all his spare time in the regions of the fourth or *nth* dimension, or in non-Euclidean space of any kind.

Moreover, the college instructor in mathematics must be ever ready to adjust himself to conditions as he finds them. He cannot afford to wait till he finds his students prepared to his ideal. He must introduce the methods in which he finds his students lacking, if he allows them to enter his classes, even if he finds it necessary to reduce the amount of work he might otherwise do. He must suggest lines of application and require actual work in applying the principles as they come up. In the very short lived journal, School Mathematics, there appeared a definition: "Mathematics is that science in which you do not know what it is you are talking about and you do not care whether what you say about it is true." Doubtless this definition was first given as a joke, and, of course, it was repeated as such, but it meets a pretty hearty response in the mind of the college man who was required, without adequate preparation, to take some courses in mathematics, which should have been elective, under an instructor who did not know much besides mathematics and hence was not a good teacher of mathematics.

In the following suggestions of a more specific character I shall deal only with methods which I have actually tried with some good results.

In college algebra I find many entering freshmen who have the ordinary preparation but know nothing about graphs. The time is well spent in teaching how to plot both linear and quadratic equations, both for the purpose of showing the general nature of functional relations and also for showing the correlation between intersections and the simultaneous values of the variables gotten by elimination. He may also understand why in some cases he gets some values which do not have corresponding intersections. Complex numbers are seen to correspond to no intersection. The plotting of complex numbers by the conventional method does not make them any more real, yet it serves the purpose of showing how real, complex and pure imaginary numbers are correlated, and ultimately leads to a more effective way of dealing with the theory of equations. If only it were possible, complex numbers might well remain out of sight; but they are always turning up in connection with equations of the second and higher degrees, and it is surely best to reduce them to an orderly classification rather than to leave them lying around where they will be stumbling blocks. They are unmanageable creatures, to say the least, but since we are obliged to associate with them, it is best to get on good terms with them as soon as possible. After all, some good things may be said about them.

Before beginning the subject of chance last year I asked each of my algebra students to get four coins and make eighty throws of these coins and record the number of times they got 4 heads, 3 heads, 2 heads, 1 head, no heads. They were given no suggestion that one of these results was more likely than another. Some individual results were surprisingly discordant with theory. I averaged the results by fives taken at random, showing that these averages agreed with each other much more nearly than did the individual results. I then took the average of the first averages. Of course, most of the students caught the reasons for the results which we found, and could state them in a very intelligent way. Some could even figure out the exact theory. The closeness of the final average with the theory was a lesson in the canceling of accidental errors in the long run. Their interest was aroused and their perception sharpened for the further study of chance and choice.

In trigonometry the field for drawing to scale and choosing practical problems is very fertile. Problems made up of actual data accumulated by the surveying class in connection with familiar objects about the campus, always arouse interest and help the student to understand how this study is useful. A frame made from four strips of wood fastened together with a specially made set screw may be used to show how the functions vary with the angle, by making the angle and the triangle, from which the functions are defined, actually grow through all of the quadrants. The triangle will vanish as one of its sides passes through 0, reappear in the next quadrant in a position which is seen to fit the original definition. A symmetrical diagram with four general statements enables the student to stow away in available form all the relations among the different functions of a single angle with no real effort to memorize. Of course this diagram must not be allowed to usurp the place of a proof any more than do Napier's rules in spherical trigonometry. But the general statements and the diagram appeal to the visual sense of symmetry, which is not only interested and pleased, but renders effective aid in future work. The Walter Smith school square No. 2 is part of the outfit of each of my students in trigonometry. It is made of heavy cardboard distinctly printed, with the means of measuring angles and tenths of inches, and is sufficiently accurate for drawing rectilinear figures to scale. It can easily be sold for 10 cents, and is the best thing for the purpose I have seen.

Spherical trigonometry can be applied to finding distance and direction to some point on the earth's surface which is attracting attention at the time. The only data needed are the latitude and longitude of the places. A few simple astronomical problems, such as the time and place of sunrise at a given time of year at a given latitude, are easily understood. The student will grasp eagerly at the task of working out the specifications for a sun dial to suit a given set of conditions.

Calculus is the paradise for the application of mathematical principles to things of interest to the student. In integral calculus conclusions follow each other in rapid succession, which if allowed to go unapplied will soon disappoint the student. In differential calculus the ingenuity of the instructor is sometimes taxed to show applications, but his efforts if properly directed never fail to attract interest. Of course, there is the whole field of maxima and minima values of functions with indefinite application to natural phenomena, but even here the full advantage of the situation is not always utilized. In the standard problem of the maximum contents of a box made from a rectangular piece of cardboard much valuable information may be brought out as to the way in which the truth of an equation applies to material things. The edge of the square cut out is naturally made the independent variable. When the graph is constructed it shows nicely the maximum value, but it also shows that only a small range of actual values of the independent variable will give a material box, while the requirements of the rest of the locus are inconsistent with the properties of matter and so do not represent a real box. Yet all the quantities are real.

With the object of showing how processes may be simplified I usually ask my students in differential calculus to develop a series of antitangents whose sum equals $\frac{\pi}{4}$ so chosen that they converge rapidly and with easy calculation. Then we actually work out the value of π to 20 decimal places, making a careful arrangement of the work to facilitate comparison. The work is pleasingly easy compared with the method suggested in most geometries, but rarely followed out even to six correct decimal places. The work would scarcely be doubled if carried out to 40 places. The student thus learns the power of the instrument he is learning to use, and also gets drill in the actual carrying out of a complicated plan to a definite conclusion with several means of facilitating computation learned along the way.

I often take up a study of the catenary curve in the midst of differential calculus. It is easy to show from a few simple principles of mechanics how the differential equations are obtained. A little anticipation of the work of integration introduces that subject under favorable auspices. The student grasps the application readily and easily and is pleased with the results. He sees the practical application to suspension bridge work, and realizes that he has found a new method of getting the equation of a curve and its length. He is easily shown that if the origin of coordinates be properly chosen the ordinates will be the measure of the tension to which the cable is subjected at any point per unit of weight in a unit of length of the cable.

As before intimated, integral calculus is full of possibilities of the most interesting and practical character when viewed from the standpoint of the student of science or engineering. I cannot dwell on details here. I usually end my elementary course in integral calculus with a consideration of force and consequent motion. The force of gravitation varying according to Newton's law is chosen. The differential equations are easily derived and the integrations easy, and the complicated formula of planetary motion are easily reached, and are always full of interest. As the student comes to the realization that the formulae for falling bodies are merely the forms which these equations assume when the acceleration is made equal to the constant g, he is not only pleased to come back to earth, but is well satisfied with his tour of investigation into the regions of space.

One text book in calculus which I used, for one year only, made what I consider the grave error of concerning itself with putting all the proofs in such a form as to make them impregnable to any attack now and forever. The object in itself was a worthy one, but the sacrifice was too great and the goal scarcely reached. The students did not appreciate the difficulties and consequently did not appreciate the *manner* of avoiding them. In most cases their work did not carry them into the region of the difficulty. Consequently the process seemed to them like putting up men of straw for the purpose of vanquishing them. It seemed to me like introducing the principles of the higher criticism to Sunday school scholars. It is sometimes best to call the student's attention to some point of attack which he may sometime be called upon to strengthen, but that he will do well to accept the results of others at that point till he has wider experience in the application of his subject.

And now, in answer to the direct question of my subject, I wish to say that many think the pendulum is swinging to the extreme in these matters.

There is danger that the bright student who sees at a flash the entire application of his results and who will feel unnecessarily restrained if obliged to go through the details of graphing and applying his equations too often. For others the available time is usually so limited that it becomes an individual question how to direct the mathematical work of each student so as to get the maximum result of mathematical training which shall be available for use in after life.